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GENERALIZED ALGORITHM TO THE EXTRACTION OF HEIGHT RIDGES IN RIEMANNIAN GEOMETRY

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ABSTRACT

Surface creases (ridges and ravines) provide us with important information about the shapes of objects and can be intuitively defined as curves on a surface along which the surface bends sharply. These features are a task in many areas such as computer vision and image processing. Even though a significant amount of research has been directed to defining and extracting ridges and ravines some fundamental challenges remain.

The authors in [6, 21] have recently shown the attraction of ridge and height ridge as a generalized local maximum in *2-D* Riemannian Geometry, and have presented a new algorithm to extract height ridges from *2-D* images. Here, we are concerned also with attraction ridge and height ridge definitions as a generalized local maximum, but in *n-D* Riemannian Geometry, and then we have a new algorithm to extract height ridges from *3-D* and n-D images. The results in this paper considered as a continuation to [1, 2, 3].

KEYWORDS: Ridges; Height Ridges; Ridge Directions

1. INTRODUCTION

Surface creases (ridges and ravines) provide us with important information about the shapes of objects and can be intuitively defined as curves on a surface along which the surface bends sharply. Mathematical description of such features is based on a study of extrema of the principal curvatures along their curvature lines. On a smooth generic surface, we define ridges to be the local positive maxima of the maximal principal curvature along its associated curvature line and ravines to be the local negative minima of the minimal principal curvature along its associated curvature line. The ridges and ravines are important for shape analysis and possess remarkable mathematical properties. These features are one of the most sought after features in areas ranging from computer vision [11, 15] and image processing [8] to tensor analysis [20, 23] and combustion simulations [10]. Consequently, defining and extracting ridges from digital data has received significant attention across different communities resulting in various competing concepts and a plethora of algorithms.

Eberly et al. [8] compare several definitions of ridges, extend Haralick's 'height' definition [12] into multiple dimensions, height ridges are a commonly used ridge structure [8,9,17,18,19].

The authors in [4,5] provided a new approach to extract ridges (as local maximum) and ravines (as local minimum) in images by the gradient, Hessian matrix and its derivatives of this images, and they provided a new approach in [18], a new algorithm in [6] to extract height ridges (generalized local maximum) on *2-D* images.

The basic concepts in linear algebra which utilized in this paper can be found in a text on matrix analysis such as [14], also, basic concepts in differential geometry (local extrema and tensors) can be found in a standard calculus text such as [16, 17, 22].

The paper is organized as follows. In the second section, we review the definitions of ridges and height ridges as a generalized local maximum in *n-D* Riemannian geometry, and *n-D* Euclidean geometry. In the third section, we provided an algorithm for extracting the height ridge on images in *3-D* Riemannian geometry and *3-D* Euclidean geometry. The fourth section we provided a generalized algorithm to extracting the height ridge on images in *n-D* Riemannian geometry and *n-D* Euclidean geometry.

2. RIDGES

In this section, we present generalized for definitions of ridges and height ridges in *n-D* Riemannian geometry, and then we review definitions of ridges and height ridges in *n-D* Euclidean geometry as a special case for *n-D* Riemannian geometry.

2.1. Ridges as Generalized Local Maximum

We now take a closer look at the definition of local Maximum of a function $f \in c^2(\mathbb{R}^n, \mathbb{R})$ Ridges will be a generalization of local maxima whereby the test for maximality of $f(x)$ is made in a restricted neighborhood of x. A similar concept of ravines generalizes local minima, but since local minima of f are local maxima of $-f$, it is sufficient to study only the concept of the ridge.

Let $f \in c^2(\mathbb{R}^n, \mathbb{R})$ where $\mathbb{R}^n$ is assigned a Riemannian geometry. A point x is a critical point for f if $v^i f_{,i}(x) = 0$ for all directions v. At a critical point, $f(x)$ is a local maximum if $v^i f_{;ij}(x)\, v^j < 0$ for all directions v. The point x is called a local maximum point. Let v_1 through v_n be linearly independent vectors and define the tensor $v^r_{.c}$ where, as a matrix, the c^{th} column is the vector v_c.

The test for local maximum points becomes $v^j_{.i} f_{,j} = 0$ for $1 \le i \le n$ and $v^k_{.i} v^l_{.j} f_{,kl}$ is an $n \times n$ negative definite tensor.

Rather than testing for a ridge (local maximum) in all n directions, It is possible to restrict attention to the only

$n - d$ directions for $d = 0$ or $d = 1$. Let the directions be denoted v_1 through v_{n-d} and define the $n \times n\text{-}d$ matrix $V = [v^r_{.c}]$ where $1 \le r \le n$ and $1 \le c \le n\text{-}d$.

Definition 2. 1. A point x is a Ridge point of type d relative to V if $v^j_{.i} f_{,j} = 0$ for $1 \le i \le n\text{-}d$ and $v^k_{.i} v^l_{.j} f_{,kl}$ is an $n\text{-}d \times n\text{-}d$ negative definite tensor.

Note that $v^j_{.i} f_{,j} = 0$ is a system of $n - d$ equations in n unknowns which, by the Implicit Function Theorem, typically has solutions which lie on *d-dimensional* manifolds. The terminology type d in the definition is used to reflect the expected dimensionality of the solution set. Also, note that local Maximum are just special cases of this definition when

$d = 0$. Such points typically are isolated and are labeled as *0-dimensional* structures.

In the case $d = 0$, any choice of V yields the same local Maximum. When $d = 1$, there are many choices for V. In this paper we will concentrate on the cases $d = 0$, $d = 1$ and only one choice for V. Generally, the choices will depend on the needs of a particular application.

Special Case

Ridges as Generalized Local Maximum in *n-D* Euclidean Geometry. A point x is a critical point for f if

$v^t Df(x) = 0$ for all directions v (equivalently, $Df(x) = 0$).

At a critical point, $f(x)$ is a local maximum if $v^t D^2 f(x) v < 0$ for all directions v (equivalently, $D^2 f(x)$ is negative definite) and in this case x is said to be a local maximum point.

Let v_1 through v_n be linearly independent directions and denote V as the $n \times n$ matrix whose columns consist of these directions. The test for ridge (local maximum) points is concisely written as $V^t\, Df(x) = 0$ and $V^t D^2 f(x) V < 0$.

Rather than testing for a ridge (local maximum) in all n directions, it is possible to restrict attention to only n-d directions for $d = 0$ or $d = 1$. The definition is summarized below.

Definition 2. 2. Let $f \in c^2(\mathbb{R}^n, \mathbb{R})$. For $d = 0$ or $d = 1$ and an $n \times n$-d matrix V of rank n-d, the point x is a Ridge point of type d with respect to V if $V^t\, Df(x) = 0$ and $V^t D^2 f(x) V < 0$.

2.2. Height Ridges

The convexity (concavity) play the essential role in the choice of the vectors V on the height ridge. Therefore, a ridge point on the graph is a place which f has a generalized local maximum in *n-dimensional* directions where the graph of f is concave. The motivation comes from the case *2-dimensional* images, where peaks of the terrain $d = 0$ can be simply characterized a local maximum. The height ridge is a ridge point for which the function has a local maximum in the direction for which the graph has the largest concavity. Since eigenvalues of $f_{,ij}$ measure convexity, and concavity in the corresponding eigendirections, a natural definition for general height ridges is given as the following

Definition 2.3. (*Height Ridge*). Let $f \in c^2(\mathbb{R}^n, \mathbb{R})$ where $\mathbb{R}^n$ is assigned a Riemannian space. Let λ_i and v_i,

$(i = 1, 2, \dots, n)$, be the generalized eigenvalues and eigenvectors respectively for $f_{,ij}$ in the following sense. Define the diagonal tensor $\lambda^r_{.c}$ whose i^{th} diagonal entry is λ_i and where $\lambda_1 \leq \lambda_{2 \leq \dots \leq} \lambda_n$. Define the tensor $v^r_{.c}$ whose c^{th} column of a matrix is the vector v_c and for which $v^k_{\ i}\, v_{kj} = \delta_{ij}$. Finally, let the tensors satisfy $f_{,ij}\, v^j_{.k} = v_{ij}\, \lambda^j_{.k}$.

A point x is a *d-dimensional* ridge point if it is a generalized local maximum point of type $d = 0$ or $d = 1$ with respect to $v^r_{.c}$. Since $v^k_{.i}\, f_{,kl}\, v^l_{.j} = \delta_{il}\, \lambda^l_{.j}$ is diagonal and since the eigenvalues are ordered, the test for a ridge point reduces to $v^j_{.i}\, f_{,j}(x) = 0$, $(i = 1, 2, \dots, n - d)$ and $\lambda_{n-d}(x) < 0$.

Special Case

The metric g_{ij} in Euclidean geometry is equal to δ_{ij}, thus, we present the special height ridge definition in *n-D* Euclidean geometry as the following

Definition 2.4. (*Height Ridge*). Let $f \in c^2(\mathbb{R}^n, \mathbb{R})$ where $\mathbb{R}^n$ is assigned a Euclidean space. Let λ_i, $(i = 1, 2, \dots, n)$, be the eigenvalues of $D^2 f$ with $\lambda_1 \leq \lambda_{2 \leq \dots \leq} \lambda_n$. Let v_i, $(i = 1, 2, \dots, n)$, be corresponding unit-length eigenvectors. A point x is a *d-dimensional* ridge point if x is a generalized local maximum point of type d with respect to $V = [v_1 \dots v_{n-d}]$. Since

$V^t D^2 f\, V = diag\{ \lambda_1, \dots, \lambda_{n-d}\}$ and since the eigenvalues are ordered, the test for a ridge point reduces to $V^t\, Df(x) = 0$ and $\lambda_{n-d}(x) < 0$.

Remark. The restriction to unit-length eigenvectors is not necessary.

3. GENERALIZED ALGORITHM FOR EXTRACTION HEIGHT RIDGES

Constructing closed-form representations for ridges is generally intractable. For *2-D* through *4-D*, it is possible to solve symbolically for the eigenvalues and eigenvectors in closed form. However, solving explicitly for the roots of the first derivative equations is not possible.

For dimensions $n \geq 5$ there are no formulas for the roots of polynomials of degree *n ([10], Theorem 5.7.3).* Numerical algorithms for solving eigensystems must be used instead. We provide ridge algorithms which lend themselves to numerical computation.

As in the case of *2-D* images [6], we construct ridges on a subpixel level by selecting an initial guess to a ridge point, searching for a nearby ridge point, then traversing the ridge curve by following its tangents.

3.1. The Generalized Algorithm in $\mathbb{R}^3$

3.1.1. The Generalized Algorithm in *3-D* Euclidean Geometry

The construction of the ridge near the semi-umbilics (at least two eigenvalues are equal) is very complicated because the eigenvectors can be discontinuous.

The eigensystems for $D^2 f$ are given by $f_{,ij}\, u_j = \alpha\, u_i$, $f_{,ij}\, v_j = \beta\, v_i$, and $f_{,ij}\, w_j = \gamma\, w_i$ where $\alpha \leq \beta \leq \gamma$ and *u*, *v*, and *w* form a right- handed orthonormal system (the vectors are all unit length, mutually orthogonal, and $e_{ijk} u_i v_j w_k = 1$). Define

$P = u_i f_{,i}$, $Q = v_i f_{,i}$, and $R = w_i f_{,i}$.

According to the height ridge definition, a point $x \in \mathbb{R}^3$ is a *1-D* ridge point if $P(x) = 0$, $Q(x) = 0$, and $\beta(x) < 0$, the two equations ($P = 0$ and $Q = 0$) define surfaces whose intersection is the ridge curve. The tangents of curves of intersecting surfaces are orthogonal to both surface normals, so the ridge direction is DP × DQ. The remainder of this section shows how to compute the ridge direction and uses tensor notation.

We assume that *f* is, at least, a C^3 function, this guarantees continuity of ridge directions, except possibly at umbilics points (this is points which $\alpha = \beta = \gamma$) or at semi-umbilics points (this is points which $\alpha = \beta$ or $\beta = \gamma$).

At semi-umbilics ($\alpha = \beta$) the eigenspace has a dimension larger than 1. Although the eigenvectors may become discontinuous, it is possible to choose a smoothly varying basis for the eigenspace. Assume that $\beta < \gamma$ in the region for which we seek ridges. Let $\bar{u}$ and $\bar{v}$ be smoothly varying orthonormal vectors which span $\langle w \rangle^{\perp}$. The eigenvector basis and the smooth basis are related by

$$\begin{bmatrix} \bar{u}_k \\ \bar{v}_k \end{bmatrix} = \begin{bmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \end{bmatrix} \begin{bmatrix} u_k \\ v_k \end{bmatrix} = C^t \begin{bmatrix} u_k \\ v_k \end{bmatrix}$$

where $C = [c_{ij}]$ is an orthogonal matrix. Define

$$\begin{bmatrix} \bar{P} \\ \bar{Q} \end{bmatrix} = C^t \begin{bmatrix} P \\ Q \end{bmatrix},$$

so $P = 0$ and $Q = 0$ if and only if $\bar{P} = 0$ and $\bar{Q} = 0$. The ridge algorithms will require differentiating $\bar{P}$ and $\bar{Q}$. The following development provides closed form solutions for these derivatives.

Since $\bar{u}$, $\bar{v}$, and w form a smoothly varying orthonormal system, their derivatives satisfy

$$\left[\begin{array}{c} \overline{u}_{i,j} \\ \overline{v}_{i,j} \\ \hline w_{i,j} \end{array}\right] = \left[\begin{array}{ccc} 0 & 0 & a_j \\ 0 & 0 & b_j \\ \hline -a_j & -b_j & 0 \end{array}\right] \left[\begin{array}{c} \overline{u}_i \\ \overline{v}_i \\ \hline w_i \end{array}\right]$$

for some choice of continuous vectors a_j and b_j.

Differentiating $\bar{P}$ and $\bar{Q}$ yields

$$\left[\begin{array}{c} \overline{P}_{,k} \\ \overline{Q}_{,k} \end{array}\right] = \left[\begin{array}{c} \overline{u}_i f_{,ik} + \overline{u}_{i,k} f_{,i} \\ \overline{v}_i f_{,ik} + \overline{v}_{i,k} f_{,i} \end{array}\right] = \left[\begin{array}{c} f_{,ki}\overline{u}_i + Ra_k \\ f_{,ki}\overline{v}_i + Rb_k \end{array}\right].$$

The two vectors a_k and b_k are determined by the eigensystem for w, namely

$$f_{,ij} w_j = \gamma w_i.$$

Differentiate this to obtain

$$f_{,ij} w_{j,k} + f_{,ijk} w_j = \gamma w_{i,k} + \gamma_{,k} w_i.$$

Substitute for $w_{i,j}$ and rearrange to obtain

$$(f_{,ij}\overline{u}_j - \gamma\overline{u}_i)a_k + (f_{,ij}\overline{v}_j - \gamma\overline{v}_i)b_k = f_{,ijk} w_j - \gamma_{,k} w_i$$

Contracting with $\bar{u}$ and $\bar{v}$ respectively yields

$$(\overline{u}_i f_{,ij}\overline{u}_j - \gamma)a_k + (\overline{u}_i f_{,ij}\overline{v}_j)b_k = f_{,ijk}\overline{u}_i w_j\,,$$
$$(\overline{v}_i f_{,ij}\overline{u}_j)a_k + (\overline{v}_i f_{,ij}\overline{v}_j - \gamma)b_k = f_{,ijk}\overline{v}_i w_j.$$

This is a system of two equations in the two unknown vectors a_k and b_k which can be solved explicitly as

$$\left[\begin{array}{c} a_k \\ b_k \end{array}\right] = \left[\begin{array}{cc} \overline{u}_i f_{,ij}\overline{u}_j - \gamma & \overline{u}_i f_{,ij}\overline{v}_j \\ \overline{v}_i f_{,ij}\overline{u}_j & \overline{v}_i f_{,ij}\overline{v}_j - \gamma \end{array}\right]^{-1} \left[\begin{array}{c} f_{,ijk}\overline{u}_i w_j \\ f_{,ijk}\overline{v}_i w_j \end{array}\right].$$

Using the relationships between the eigenvectors and the smooth basis, we obtain

$$\begin{bmatrix} a_k \\ b_k \end{bmatrix} = C^t \begin{bmatrix} \frac{1}{\alpha-\gamma} f_{,ijk} u_i w_j \\ \frac{1}{\beta-\gamma} f_{,ijk} v_i w_j \end{bmatrix}.$$

Substituting into the formulas for $\bar{P}_{,k}$ and $\bar{Q}_{,k}$ yields

$$\begin{bmatrix} \overline{P}_{,k} \\ \overline{Q}_{,k} \end{bmatrix} = C^t \begin{bmatrix} \alpha u_k + \frac{R}{\alpha-\gamma} f_{,ijk} u_i w_j \\ \beta v_k + \frac{R}{\beta-\gamma} f_{,ijk} v_i w_j \end{bmatrix}$$

The matrix *C* is an unknown quantity, but we will see that ridge traversal only requires knowing $det(C) = \pm 1$. With this in mind, define

$$\begin{bmatrix} \widetilde{P}_{,k} \\ \widetilde{Q}_{,k} \end{bmatrix} = \begin{bmatrix} \alpha u_k + \frac{R}{\alpha-\gamma} f_{,ijk} u_i w_j \\ \beta v_k + \frac{R}{\beta-\gamma} f_{,ijk} v_i w_j \end{bmatrix} \tag{3.1}$$

These quantities will in effect play the role of the derivatives of $\bar{P}$ and $\bar{Q}$ despite the fact that they are not necessarily the derivatives of some functions $\tilde{P}$ and $\tilde{Q}$. The use of index/derivative notation is used suggestively to remind us how the quantities were obtained.

Ridge Flow

Given an initial approximation $\mathcal{A}$ to a ridge point, a flow path to the ridge is determined by gradient descent. Ridge points occur as absolute minimum points for the function $(\bar{P}^2(x) + \bar{Q}^2(x))/2$. Note that $[\bar{P} \;\; \bar{Q}] = [P \;\; Q]C$ and $[\bar{P}_{,k} \;\; \bar{Q}_{,k}] = [\tilde{P}_{,k} \;\; \tilde{Q}_{,k}]C$, so $\bar{P}\bar{P}_{,i} + \bar{Q}\bar{Q}_{,i} = P\tilde{P}_{,k} + Q\tilde{Q}_{,k}$.

Therefore, the gradient descent is modeled by

$$\frac{dx_i(t)}{dt} = -P(x(t))\widetilde{P}_{,i}(x(t)) - Q(x(t))\widetilde{Q}_{,i}(x(t)), \quad x_i(0) = \mathcal{A}_i,\ i = 1, 2, 3. \tag{3.2}$$

The solution curve terminates at time $T > 0$ if $P(x(T)) = 0$ and $Q(x(T)) = 0$, or if a positive local minimum is reached, in which case a different starting point should be used. The point $\mathcal{R} = x(T)$ will be used as the starting ridge point for ridge traversal. The nice consequence of this result is that one does not have to explicitly construct smoothly varying $\bar{u}$ and $\bar{v}$ in order to compute the flow direction to a ridge. The (possibly discontinuous) eigenvectors *u* and *v* can be used instead to produce the continuous flow direction.

Ridge Traversal

Let $\mathcal{R}$ be the initial ridge point obtained by ridge flow. If $T(x)$ is a unit length tangent vector to the ridge, then the ridge can be traversed by solving a system of ordinary differential equations, $dx/dt = T(x)$. To determine $T(x)$, note that the ridge curve is a solution to $\bar{P}(x) = 0$ and $\bar{Q}(x) = 0$. The curve is the intersection of the two implicitly defined surfaces whose normals are $D\bar{P}$ and $D\bar{Q}$. The curve direction must be perpendicular to both normals, so choose T to be the cross product of the normals, $T_i(x) = e_{ijk}\bar{P}_{,j}(x)\bar{Q}_{,k}(x)$ where e_{ijk} is the permutation tensor on three symbols.

As in ridge flow, we do not have to explicitly construct smoothly varying $\bar{u}$ and $\bar{v}$ to obtain a continuous ridge direction.

Since $[D\bar{P}\ \ D\bar{Q}] = [D\tilde{P}\ \ D\tilde{Q}]C$ and C is orthogonal, the cross products are related by $D\bar{P}(x) \times D\bar{Q}(x) = det\big(C(x)\big) D\tilde{P}(x) \times D\tilde{Q}(x)$, where $|det(C(x))| = 1$. The discontinuity of the cross product is captured entirely by $det(C(x))$. If a semi-umbilic $\alpha=\beta$ causes eigenspaces to be swapped, or if the numerical eigensolver does not provide a smoothly varying set of eigenvectors as x varies, then such behaviour will affect $det(C(x))$ and can be detected in the implementation by comparing the angle between the previously computed direction and the currently computed direction. The system of equations determining the traversal is therefore

$$\frac{dx_i(t)}{dt} = \pm e_{ijk}\widetilde{P}_{,j}(x(t))\widetilde{Q}_{,k}(x(t)), \quad x_i(0) = \mathcal{R}_i,\ i = 1,2,3. \qquad (3.3)$$

where the two traversals are required.

3.1.2. The Generalized Algorithm in *3-D* Riemannian Geometry

The height ridge definition which was discussed in detail in section *(3.1.1)* is extended to the case of Riemannian geometry. Let $f_{,ij}\, u^j = \alpha\, u_i$, $f_{,ij}\, v^j = \beta\, v_i$, and $f_{,ij}\, w^j = \gamma\, w_i$ where $\alpha \le \beta \le \gamma$, $u_i u^i = v_i v^i = w_i w^i = 1$, and $u_i v^i = u_i w^i = v_i w^i = 0$. Define $P = u^i f_{,i}$, $Q = v^i f_{,i}$, and $R = w^i f_{,i}$. According to the height ridge definition, a point $x \in \mathbb{R}^3$ is a *1-dimensional* ridge point if $P(x) = 0$, $Q(x) = 0$, and $\beta(x) < 0$.

Equation *(3.1)* generalizes to

$$\begin{bmatrix} \widetilde{P}_{,k} \\ \widetilde{Q}_{,k} \end{bmatrix} = \begin{bmatrix} \alpha u_k + \frac{R}{\alpha-\gamma} f_{,ijk} u^i w^j \\ \beta v_k + \frac{R}{\beta-\gamma} f_{,ijk} v^i w^j \end{bmatrix} \qquad (3.4)$$

which specifies the quantities that play the role of the covariant derivatives of P and Q despite the fact that neither $\tilde{P}_{,k}$ nor $\tilde{Q}_{,k}$ are the covariant derivatives of some tensors $\tilde{P}$ or $\tilde{Q}$. The model for ridge flow given by equation *(3.2)* generalizes to

$$\frac{dx^i}{dt} = -g^{ij}(P\widetilde{P}_{,j} + Q\widetilde{Q}_{,j}) = -P\widetilde{P}^i_{,,} - Q\widetilde{Q}^i_{,,},\ x^i(0) = \mathcal{A}^i,\ i = 1,2,3, \tag{3.5}$$

where $\mathcal{A}$ is an initial approximation to the ridge. If the flow terminates at time $T > 0$ where $P(x(T)) = 0$, then $\mathcal{R}$ = x(T) is used as the starting ridge point for ridge traversal. Ridge traversal given by equation *(3.3)* generalizes to

$$\frac{dx^i}{dt} = \pm g^{ij} e_{jkl} g^{kr} g^{ls} \widetilde{P}_{,r}\widetilde{Q}_{,s} = \pm e^{ijk}\widetilde{P}_{,j}\widetilde{Q}_{,k}, \quad x^i(0) = \mathcal{R}^i(0),\ i = 1,2,3, \tag{3.6}$$

where e_{ijk} is the permutation tensor on three symbols. the raising of the indices on the permutation tensor is required to produce a tangent vector dx^i/dt.

3.2. The Generalized Algorithm in $\mathbb{R}^n$

3.2.1. The Generalized Algorithm in *n-D* Euclidean Geometry

The problems with semi-umbilics occur for $n \geq 3$. The ideas for dimension $n=3$ generalize to higher dimensions. Let the eigenvalues and eigenvectors for $D^2 f$ be denoted respectively λ_k and v_k for $1 \leq k \leq n$. The semi-umbilics $\lambda_k = \lambda_n$ correspond to branch points and end points of the ridge. The semi-umbilics $\lambda_i = \lambda_k$j $(i \neq n, j \neq n)$ reflect symmetries of the dataset, but should not cause termination of the ridge construction. We assume that the ridge construction is in regions where $\lambda_{n-1} < \lambda_n$ so that the eigenspaces corresponding to the first *n-1* eigenvectors never swap or combine with the eigenspace corresponding to the last eigenvalue λ_n.

Let v_{ij} be an orthonormal matrix (with determinant 1) whose columns are unit-length eigenvectors for $D^2 f$; the columns form a right-handed orthonormal system. All the eigensystems can be written as a single matrix equation

$f_{,ij}\ v_{jk} = v_{ij}\ \lambda_{jk}$ where λ_{ii} is a diagonal matrix whose j^{th} diagonal entry is the eigenvalue corresponding to the eigenvector which is the j^{th} column of v_{ij}.

Define $P_j = v_{ij} f_{,i}$. According to the height ridge definition, a point $x \in \mathbb{R}^n$ is a *1-dimensional* ridge point if $P_j(x) = 0$ for $1 \leq j \leq n-1$ and $\lambda_{n-1}(x) < 0$, the *n-1* equations $P_j(x) = 0$ define *n-1* hypersurfaces whose intersection is the ridge curve. The ridge directions must be orthogonal to all the hypersurface normals, so it is the generalized cross product of the gradients DP_j. The remainder of this subsection shows how to compute the ridge direction use thing the tensor notations.

The construction includes indices whose range is between *1* and *n-1* rather than the full range between *1* and *n*. As a notational aid, indices in the range *1* through *n-1* will be subscripted with a *zero*. Indices in the full range are unsubscripted. The index *n* is the dimension of space and does not indicate a free index.

Define $w_i = v_{in}$, $\gamma = \lambda_n$ and $R = w_i f_{,i}$. Let $\overline{v}_{ij}$ denote a smoothly varying orthonormal matrix whose last column is w and whose first *n-1* columns span $\prec w \succ^\perp$. The first *n-1* eigenvectors and the smooth basis for $\prec w \succ^\perp$ are related by

$$\overline{v}_{ii_0} = v_{ij_0} c_{j_0 i_0}$$

where $C_{i_0 j_0}$ is an orthogonal matrix, define

$$\overline{P}_{i_0} = \overline{v}_{ii_0} f_{,i} = P_{j_0} c_{j_0 i_0}$$

so $\overline{P}_{i_0} = 0$ if and only if $P_{i_0} = 0$. The ridge algorithms will require differentiating $\overline{P}_{i_0}$.

Orthonormality of $\overline{v}_{ij}$ implies

$$\overline{v}_{ki}\overline{v}_{kj} = \delta_{ij} = \overline{v}_{ik}\overline{v}_{jk}$$

differentiating yields

$$\overline{v}_{ki}\overline{v}_{kj,m} + \overline{v}_{ki,m}\overline{v}_{kj} = 0.$$

The derivatives $\overline{v}_{kj,m}$ can be written in terms of the orthonormal basis as

$$\overline{v}_{kj,m} = a_{jlm}\overline{v}_{kl}$$

where a_{jlm} is a continuous quantity that will be determined later. Replacing this in the previous equation yields

$$0 = \overline{v}_{ki} a_{jlm} \overline{v}_{kl} + a_{ilm}\overline{v}_{kl}\overline{v}_{kj} = a_{jim} + a_{ijm}.$$

Thus, a_{jlm} is antisymmetric in its first two indices. Without loss of generality, we can choose $a_{i_0 j_0 k} = 0$,

$(i_0 \neq j_0)$ since these components represent rotations of the vectors within the orthogonal complement of w. The vectors must be a solution to

$$\overline{v}_{ii_0,j} = a_{i_0 nj} w_i \quad and \quad w_{i,j} = -a_{i_0 nj}\overline{v}_{ii_0}$$

differentiating $\overline{P}_{i_0}$ yields

$$\overline{P}_{i_0,k} = \overline{v}_{ii_0} f_{,ik} + \overline{v}_{ii_0,k} f_{,i} = f_{,ki}\overline{v}_{ii_0} + R a_{i_0 nk}.$$

The $a_{i_0 nk}$ are determined by the eigensystem for w, namely

$$f_{,ij} w_j = \gamma w_i$$

differentiate this to obtain

$$f_{,ij} w_{j,k} + f_{,ijk} w_j = \gamma w_{i,k} + \gamma_{,k} w_i$$

substitute for $w_{i,j}$ and rearrange to obtain

$$(f_{,ij}\overline{v}_{ji_0} - \gamma\overline{v}_{ii_0})a_{i_0nk} = f_{,ijk}w_j - \gamma_{,k}w_i$$

contracting with $\overline{v}_{ij_0}$ yields

$$(\overline{v}_{ij_0}f_{,ij}\overline{v}_{ji_0} - \gamma\delta_{i_0j_0})a_{i_0nk} = f_{,ijk}\overline{v}_{ij_0}w_j$$

This system of equations can be solved explicitly as follows. Using the relationships between the eigenvectors and the smooth basis, we have

$$\begin{aligned}
\overline{v}_{ij_0}f_{,ij}\overline{v}_{ji_0} - \gamma\delta_{j_0i_0} &= v_{ik_0}c_{k_0j_0}f_{,ij}v_{jm_0}c_{l_0i_0} - \gamma\delta_{j_0i_0} \\
&= c_{k_0j_0}(v_{ik_0}f_{,ij}v_{jm_0})c_{m_0i_0} - \gamma\delta_{j_0i_0} \\
&= c_{k_0j_0}(v_{ik_0}v_{ij}\lambda_{jm_0})c_{m_0i_0} - \gamma\delta_{j_0i_0} \\
&= c_{k_0j_0}(\delta_{k_0j}\lambda_{jm_0})c_{m_0i_0} - \gamma\delta_{j_0i_0} \\
&= c_{k_0j_0}\lambda_{k_0m_0}c_{m_0i_0} - \gamma c_{k_0j_0}\delta_{k_0m_0}c_{m_0i_0} \\
&= c_{k_0j_0}(\lambda_{k_0m_0} - \gamma\delta_{k_0m_0})c_{m_0i_0}
\end{aligned}$$

and

$$f_{,ijk}\overline{v}_{ij_0}w_j = c_{i_0j_0}f_{,ijk}v_{ii_0}w_j.$$

The system can be reduced as shown where $\Delta_{i_0j_0}$ is the diagonal matrix whose k_0^{th} diagonal entry is $\lambda_{k_0} - \gamma$ and $\Delta^{-1}_{i_0j_0}$ is the diagonal inverse matrix:

$$\begin{aligned}
(\overline{v}_{ij_0}f_{,ij}\overline{v}_{ji_0} - \gamma\delta_{i_0j_0})a_{i_0nk} &= f_{,ijk}\overline{v}_{ij_0}w_j \\
c_{k_0j_0}(\lambda_{k_0m_0} - \gamma\delta_{k_0m_0})c_{m_0i_0}a_{i_0nk} &= c_{i_0j_0}f_{,ijk}v_{ii_0}w_j \\
(\lambda_{j_0m_0} - \gamma\delta_{j_0m_0})c_{m_0i_0}a_{i_0nk} &= f_{,ijk}v_{ij_0}w_j \\
c_{m_0i_0}a_{i_0nk} &= \Delta^{-1}_{m_0j_0}f_{,ijk}v_{ij_0}w_j \\
a_{i_0nk} &= c_{m_0i_0}\Delta^{-1}_{m_0j_0}f_{,ijk}v_{ij_0}w_j.
\end{aligned}$$

Substituting into the formula for $\overline{P}_{\alpha,k}$ yields

$$\begin{aligned}\overline{P}_{i_0,k} &= \overline{v}_{ii_0} f_{,ik} + Ra_{i_0nk} \\ &= v_{ij_0} c_{j_0i_0} f_{,ik} + Rc_{m_0i_0} \Delta^{-1}_{m_0j_0} f_{,ijk} v_{ik_0} w_j \\ &= c_{j_0i_0}(v_{ij_0} f_{,ik} + R\Delta^{-1}_{j_0k_0} f_{,ijk} v_{ik_0} w_j) \\ &= c_{j_0i_0}(v_{kk_0} \lambda_{k_0j_0} + R\Delta^{-1}_{k_0j_0} f_{,ijk} v_{ik_0} w_j).\end{aligned}$$

Define

$$\widetilde{p}_{i_0,k} = v_{kj_0} \lambda_{j_0i_0} + R\Delta^{-1}_{i_0j_0} f_{,ijk} v_{ij_0} w_j. \tag{3.7}$$

These quantities play the role of the derivatives of $\overline{P}_{i_0}$ despite the fact that they are not necessarily the derivatives of some functions $\tilde{P}_{i_0}$.

Ridge Flow

Given an initial approximation $\mathcal{A}$ to a ridge point, a flow path to the ridge is determined by gradient descent. Ridge points occur as absolute minimum points for the function $P_{i_0}(x)P_{i_0}(x)/2$. Note that $\overline{P}_{i_0} = P_{j_0} c_{j_0 i_0}$ and $\overline{P}_{i_0,k} = c_{j_0 i_0} \tilde{P}_{j_0,k}$, so $\overline{P}_{i_0} \overline{P}_{i_0,k} = P_{i_0} \tilde{P}_{i_0,k}$. Therefore, the gradient descent is modeled by

$$\frac{dx_i(t)}{dt} = -P_{i_0}(x(t))\widetilde{P}_{i_0,i}(x(t)), \quad x_i(0) = \mathcal{A}_i,\, 1 \le i \le n. \tag{3.8}$$

The solution curve terminates at time $T > 0$ if $P_{i_0}(x(T)) = 0$ or if a positive local minimum is reached, in which case a different starting point should be used. The point $\mathcal{R} = x(T)$ will be used as the starting ridge point for ridge traversal. As in the case of *1-dimensional* ridges in $\mathbb{R}^3$, the continuous flow direction is calculated directly from (possibly discontinuous) eigenvectors and eigenvalues.

- **Ridge Traversal**

Let $\mathcal{R}$ be the initial ridge point obtained by ridge flow. If $T(x)$ is a unit length tangent vector to the ridge, then the ridge can be traversed by solving a system of ordinary differential equations, $dx/dt = T(x)$. To determine $T(x)$, note that the ridge curve is the intersection of *n-1* surfaces implicitly defined by $\overline{P}_{i_0}(x) = 0$. The curve direction is, therefore, a vector which is orthogonal to all *n-1* surface normals $P_{i_0,i}$. The way to construct a vector $T \in \mathbb{R}^n$ which is orthogonal to

n- 1 orthogonal vectors is to use the generalized cross product

$$T_i = e_{ii_1\cdots i_{n-1}} \overline{P}_{1,i_1} \cdots \overline{P}_{n-1,i_{n-1}}$$

where $e_{ii_1 \ldots i_{n-1}}$ is the permutation tensor on *n* symbols. Let $\varepsilon_{j_1 \ldots j_{n-1}}$ be the permutation tensor on *n-1* symbols; then

$$\begin{aligned} T_i &= \frac{1}{(n-1)!} e_{ii_1\cdots i_{n-1}} \varepsilon_{j_1\cdots j_{n-1}} \overline{P}_{j_1,i_1} \cdots \overline{P}_{j_{n-1},i_{n-1}} \\ &= \frac{1}{(n-1)!} e_{ii_1\cdots i_{n-1}} \varepsilon_{j_1\cdots j_{n-1}} c_{k_1 j_1} \widetilde{P}_{k_1 i_1} \cdots c_{k_{n-1} j_{n-1}} \widetilde{P}_{j_{n-1} i_{n-1}} \\ &= (\varepsilon_{j_1\cdots j_{n-1}} c_{k_1 j_1} \cdots c_{k_{n-1} j_{n-1}}) \frac{1}{(n-1)!} e_{ii_1\cdots i_{n-1}} \widetilde{P}_{k_1 i_1} \cdots \widetilde{P}_{k_{n-1} i_{n-1}} \\ &= det(C) \varepsilon_{k_1\cdots k_{n-1}} \frac{1}{(n-1)!} e_{ii_1\cdots i_{n-1}} \widetilde{P}_{k_1 i_1} \cdots \widetilde{P}_{k_{n-1} i_{n-1}} \\ &= det(C) e_{ii_1\cdots i_{n-1}} \widetilde{P}_{1 i_1} \cdots \widetilde{P}_{n-1 i_{n-1}}. \end{aligned}$$

Any sign-changes or swapping of eigenvectors within $\prec w \succ^{\perp}$ Produced by partial umbilics or the numerical eigensolver will be reflected in the determinant *det(C)*. But since *C* is orthonormal, the determinant is either *1* or *-1*. The key result again is that you do not need to keep track of smoothly varying eigenfields. One can simply compute the eigenvectors, compute the ridge direction, and choose the sign of the direction so that the current direction forms the smallest angle with the previous direction. The system of equations determining the traversal is therefore

$$\frac{dx_i(t)}{dt} = \pm e_{ii_1 \ldots i_{n-1}} \widetilde{P}_{1\, i_1}(x(t)) \ldots \widetilde{P}_{n-1\, i_{n-1}}(x(t)), \quad x_i(0) = \mathcal{R}_i,\ 1 \le i \le n. \tag{3.9}$$

where the two traversals are required.

3.2.2. The Generalized Algorithm in n-D Riemannian Geometry

The general height ridge definition for *1-dimensional* ridges which was discussed in detail in section *(3.2.1)* is extended to the case of Riemannian geometry. Let $f_{,ij}\, v^j_{.k} = v^i_{.j} \lambda^j_{.k}$ where λ is a diagonal tensor whose j^{th} diagonal entry is the generalized eigenvalue corresponding to the generalized eigenvector which is the j^{th} column of $v^i_{.j}$ treated as an $n \times n$ matrix.

Define $P_j = v^i_{.j}\, f_{,i}$. and according to the height ridge definition, a point $x \in \mathbb{R}^n$ is a *1-dimensional* ridge point if $P_j(x) = 0$ for

$1 \le j \le n - 1$ and $\lambda_{n-1}(x) < 0$.

Using the same convention for index notation as in subsection *(3.2.1)*, we can extend the equations in that section. Equation *(3.7)* generalizes to

$$\widetilde{p}_{i_0,k} = v_{kj_0} \lambda^{j_0}_{i_0} + R \Delta^{-1}_{i_0 j_0} f_{,ijk} v^i_{j_0} w^j. \tag{3.10}$$

where $\Delta^{-1}_{i_0 j_0}$ is the diagonal inverse tensor for the diagonal tensor $\Delta_{i_0 j_0}$ whose k_0^{th} diagonal entry is $\lambda_{k_0} - \gamma$. The vector *w* is the generalized eigenvector corresponding to eigenvalue $\gamma = \lambda_n$ and the value $R = w^i f_{,i}$. The summation over j_0 is not intended as a tensor summation, but is a simple arithmetic sum, so the rule for pairing a contravariant and a covariant index does not apply here. The model for ridge flow given by equation *(3.8)* generalizes to

$$\frac{dx^i}{dt} = -g^{ij} P_{i_0} \widetilde{P}_{i_0,j} = -P_{i_0} \widetilde{P}^i_{i_0,\cdot},\ x^i(0) = \mathcal{A}^i,\ 1 \le i \le n \tag{3.11}$$

where $\mathcal{A}$ is an initial approximation to the ridge. If the flow terminates at time $T > 0$ where $P(x(T)) = 0$, then $\mathcal{R} = x(T)$ is used as the starting ridge point for ridge traversal. The summation over i_0 is not intended as a tensor summation, but is just a simple arithmetic one, so the index convention of pairing a covariant with a contravariant index does not apply here. Finally, ridge traversal in equation *(3.9)* generalizes to

$$\begin{aligned} \frac{dx^i}{dt} &= \pm\, g^{ij} e_{jj_1 \ldots j_{n-1}} g^{j_1 i_1} \widetilde{P}_{1 i_1} \ldots g^{j_{n-1} i_{n-1}} \widetilde{P}_{n-1\, i_{n-1}} \\ &= \pm\, e^{i i_1 \ldots i_{n-1}} \widetilde{P}^{i_1}_{1\cdot} \ldots \widetilde{P}^{i_{n-1}}_{n-1\cdot}, \quad x^i(0) = \mathcal{R}_i,\ 1 \le i \le n. \end{aligned} \tag{3.12}$$

where $e^{i_1 \ldots i_n} = g^{i_1 j_1} \ldots g^{i_n j_n} e_{j_1 \ldots j_n}$ and $e_{j_1 \ldots j_n}$ are the permutation tensors on *n* symbols. the raising of the indices on the permutation tensor is necessary to guarantee that dx^i/dt is a tangent vector.

CONCLUSIONS

The two algorithms in this paper confirmed the results in the *2-dimensional* images in [6, 21]. For details, the ridges on the two images (saddles and translate) can be obtained using the algorithm in section 3 in [6].

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